JOURNAL OF APPROXIMATION THEORY 3, 183–193 (1970)

Necessary and Sufficient Conditions for a Compact Convex Set to be a Set of Best Approximations

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Received August 26, 1968

1. X will be a compact metric space, and $C(X)$, the space of real valued continuous functions on X. For $g(x) \in C(X)$, we define

$$\| g(x)\| = \max_{x \in X} | g(x)|.$$

Approximation of functions in $C(X)$ will be by linear combinations of n *given*, linearly independent continuous functions, $f_1(x), f_2(x),..., f_n(x)$. A linear combination of the $f_i(x)$ will be represented by $a \cdot F(x)$, where $a \in R^n$ and $F(x) = [f_1(x), f_2(x),..., f_n(x)]$. Thus $a \cdot F(x)$ is the usual dot product. Also, if $a \in R^n$, $| a |$ will denote $(a \cdot a)^{1/2}$.

For a given $g(x) \in C(X)$, the infimum of $\| a \cdot F(x) - g(x)\|$ as a ranges over R^n will be denoted by $N_c{}^*(g)$.

In the parameter space, R^n, the set of best approximations, $(B.A.)_g$, for $g(x)$, is defined by

$$(B.A.)_g = [a \in R^n : \| a \cdot F(x) - g(x)\| = N_c{}^*(g)].$$

It is known that $(B.A.)_g$ is a nonempty compact convex set. This suggests an inverse problem which we will solve. But first,

DEFINITION 1.1. A set $S \subset R^n$ is called *a set of best approximations* if there exists a $g(x) \in C(X)$ such that $S = (B.A.)_g$.

PROBLEM 1.2. Given a compact convex set $S \subset R^n$, what conditions, imposed on $F(x)$, are necessary and sufficient for S to be a set of best approximations.

In the process of solving problem (1.2), we will solve a slightly more general problem. It is

PROBLEM 1.3. Given a compact convex set S and a constant K, what conditions on $F(x)$ are necessary and sufficient to insure that there exists $g(x) \in C(X)$ such that

(1) $S = (B.A.)_g$ and

(2) $N_c^*(g) = K$.

First we will show why a solution to problem 1.3 yields a solution to problem 1.2.

To do this, we will first show that if $S = (B.A.)_g$, then $N_c^*(g) \geqslant N_c^m$ (to be defined). Then we will show that if $S = (B.A.)_g$, there exists a function $g_m(x) \in C(X)$ such that (1) $S = (B.A.)_{g_m}$ and (2) $N_c^*(g) = N_c^m$. This will prove that S is a set of best approximations if, and only if, there exists $g(x) \in C(X)$ such that $S = (B.A.)_g$ and $N_c^*(g) = N_c^m$.

To begin, let

$$N_c^m = \sup_{x \in X} [\tfrac{1}{2}[\max_{a \in S} a \cdot F(x) - \min_{a \in S} a \cdot F(x)]].$$

LEMMA 1.4. *If* $S = (B.A.)_g$, *then* $N_c^*(g) \geqslant N_c^m$.

Proof. For fixed x and all $a \in S$, $-N_c^*(g) \leqslant g(x) - a \cdot F(x) \leqslant N_c^*(g)$. Thus, $-N_c^*(g) + \max_{a \in S} a \cdot F(x) \leqslant g(x) \leqslant N_c^*(g) + \min_{a \in S} a \cdot F(x)$. Hence, for each $x \in X$, $N_c^*(g) \geqslant \frac{1}{2}(\max_{a \in S} a \cdot F(x) - \min_{a \in S} a \cdot F(x))$. Thus, $N_c^*(g) \geqslant N_c^m$.

LEMMA 1.5. $\max_{a \in S} a \cdot F(x)$ and $\min_{a \in S} a \cdot F(x)$ are both contained in $C(X)$.

Proof. Because X is compact, $F(x)$ is uniformly continuous on X. Thus, given $\epsilon > 0$, there exists a ∂ such that $|a| \cdot |F(x) - F(y)| < \epsilon$ whenever $|x - y| < \partial$ and $a \in S$.

Now, if $|x - y| < \partial$, $|a \cdot F(x) - a \cdot F(y)| \leqslant |a| \cdot |F(x) - F(y)| < \epsilon$. Hence,

$$\max_{a \in S} a \cdot F(x) > \max_{a \in S} a \cdot F(y) - \epsilon \tag{1}$$

and

$$\max_{a \in S} a \cdot F(y) > \max_{a \in S} a \cdot F(x) - \epsilon. \tag{2}$$

Therefore, $|\max_{a \in S} a \cdot F(x) - \max_{a \in S} a \cdot F(y)| < \epsilon$. The proof is identical for $\min_{a \in S} a \cdot F(x)$.

THEOREM 1.6. If $S = (B.A.)_g$, there is a function $g_m(x) \in C(X)$ such that (1) $S = (B.A.)_{g_m}$ and (2) $N_c^*(g_m) = N_c^m$.

Proof. Let

$$T_1 = [x \in X: g(x) \geqslant -N_c^m + \max_{a \in S} a \cdot F(x)],$$

$$T_2 = [x \in X: g(x) \leqslant N_c^m + \min_{a \in S} a \cdot F(x)].$$

T_1 and T_2 are closed subsets of X. Further, since

$$N_c^m + \min_{a \in S} a \cdot F(x) \geqslant -N_c^m + \max_{a \in S} a \cdot F(x),$$

we have $T_1 \cup T_2 = X$.

Define $g_m(x)$ as follows:

$$g_m(x) = \begin{cases} \min[g(x), & N_c^m + \min_{a \in S} a \cdot F(x)], \quad \text{for } x \in T_1, \\ \max[g(x), & -N_c^m + \max_{a \in S} a \cdot F(x)], \quad \text{for } x \in T_2. \end{cases}$$

$g_m(x)$ is well defined as $g(x)$ on $T_1 \cap T_2$, and it is easily verified that $g_m(x) \in C(X)$.

It can be seen that $\| g_m(x) - g(x)\| \leqslant N_c^*(g) - N_c^m$. Further, $-N_c^m + \max_{a \in S} a \cdot F(x) \leqslant g_m(x) \leqslant N_c^m + \min_{a \in S} a \cdot F(x)$, and, hence, if $a \in S$, $\| g_m(x) - a \cdot F(x)\| \leqslant N_c^m$.

If $a \notin S$, there exists an $x_a \in X$ such that $| g(x_a) - a \cdot F(x_a)| > N_c^*(g)$. Thus $| g_m(x_a) - a \cdot F(x_a)| = | g_m(x_a) - g(x_a) + g(x_a) - a \cdot F(x_a)| \geqslant | g(x_a) - a \cdot F(x_a)| - | g_m(x_a) - g(x_a)| > N_c^*(g) - (N_c^*(g) - N_c^m) = N_c^m$. Thus, if $a \notin S$, $\| g_m(x) - a \cdot F(x)\| > N_c^m$.

If $a \in S$, there exists an x_a such that $| g(x_a) - a \cdot F(x_a)| = N_c^*(g)$, and an argument similar to the one above shows that, in this case, $| g_m(x_a) - a \cdot F(x_a)| = N_c^m$. This completes the proof.

2. S will always stand for a compact convex set in R^n. If S contains the origin, we will let $L(S)$ denote the smallest linear space containing S.

It is known that S has a nonempty interior relative to $L(S)$. From now on, for convenience, we will assume that 0 is an interior point of S relative to $L(S)$. The justification for this assumption lies in the fact that the property of being a set of best approximations is maintained by translation:

LEMMA 2.1. *$S = (B.A.)_g$ and $N_c^*(g) = K$ if, and only if,*

$$S + a^* = (B.A.)_{g+a^* \cdot F} \quad \textit{and} \quad N_c^*(g + a^* \cdot F) = K.$$

Proof. This is a direct consequence of the equality

$$\| a \cdot F(x) - g(x)\| = \|(a + a^*) \cdot F(x) - (g(x) + a^* \cdot F(x))\|.$$

If a^* is a boundary point of S relative to $L(S)$, $b \in L(S)$, $b \neq 0$, and $b \cdot (a - a^*) \geqslant 0$ for all $a \in S$, then $b \cdot (a - a^*) = 0$ is called a support hyperplane to S in $L(S)$.

If $L(S) \neq 0$, it is known that through each such boundary point, a^*, there does pass such a support hyperplane.

From this it is clear that if $L(S) \neq 0$, there is a $b \in L(S)$ such that $| b | = 1$ and such that $b \cdot (a - a^*) \geqslant 0$ for all $a \in S$. Such a b will be called an inward unit normal.

At each boundary point of S relative to $L(S)$ choose a unit inward normal Call the set of these inward normals a *container of S relative to* $L(S)$. See Figs. 1, 2 for a vector interpretation. If $L(S) = 0$, define the empty set to be the only container of S relative to $L(S)$.

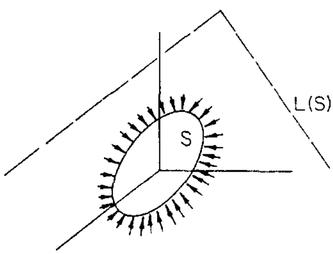

FIGURE 1

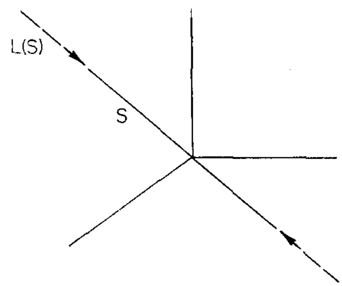

FIGURE 2

DEFINITION 2.2. A set, $C(S)$, in R^n is *a container of S* if it is the union of a container of S relative to $L(S)$ and the set of all unit vectors in $L(S)^\perp$.

Notation.

1. $L(S, b)$: The smallest linear space containing S and b;
   $L(b)$: The smallest linear space containing b.
2. $F^S(x)$: The projection of $F(x)$ on $L(S)$;
   $F^{S,b}(x)$: The projection on $L(S, b)$;
   $F^b(x)$: The projection on $L(b)$.
   Similarly, if $T \in R^n$, the respective projections are T^S, $T^{S,b}$, T^b.
3. If $\tau \subset R^n$, $\tau_u^{S,b}$ will denote the set

$$\left[\frac{T^{S,b}}{| T^{S,b} |} : T \in \tau,\ T^{S,b} \neq 0\right].$$

4. $\bar{P}$: The closure of P;
   $\mathscr{C}P$: The complement of P.

5. $F(Q)$: The image of Q under F;

6. $E_F(S) = [x \in X : \frac{1}{2}(\max_{a \in S} a \cdot F(x) - \min_{a \in S} a \cdot F(x)) = N_c^m]$.

If $\tau \subset R^n$, to express the fact that there is a container of S in the set $[(T/|T|) : T \in \tau, T \neq 0]$, one might say that τ enfolds S. More generally:

DEFINITION 2.3. A set $\tau \subset R^n$ is said to *enfold* S if there exists a container, $C(S)$, of S such that $b \in \overline{\tau_u^{S,b}}$ for all $b \in C(S)$.

3. Let S be a compact convex set containing 0 as an interior point relative to $L(S)$, and let $K \geqslant N_c^m$.

THEOREM 3.1. *A necessary and sufficient condition for the existence of* $g(x) \in C(X)$ *such that*

1) $S = (B.A.)_g$ *and*

2) $N_c^*(g) = K$

is that there exist two closed sets Q_1, Q_2, *in* X, *with the following properties*:

(i) $Q_1 \cap Q_2 = \varnothing$ *if* $K > N_c^m$,
$Q_1 \cap Q_2 = E_F(S)$ *if* $K = N_c^m$.

(ii) *There exists an* $x_0 \in Q_1 \cup Q_2$ *such that* $F(x_0) \in L(S)^\perp$.

(iii) $F(Q_1) \cup (-F(Q_2))$ *enfolds* S.

First, we prove:

LEMMA 3.2. *If* $0 \in S$ *and* $S = (B.A.)_g$, *there is an* $x_0 \in X$ *such that* $F(x_0) \in L(S)^\perp$ *and* $|g(x_0)| = N_c^*(g)$.

Proof. Let $\{a_1, a_2, ..., a_n \cdots\}$ be a countable dense subset of S. Since S is compact and convex, it is easily shown that $\sum_{n=1}^\infty a_n/2^n \in S$. Thus there is an $x_0 \in X$ such that

$$\left| \left(\sum_{n=1}^\infty \frac{a_n}{2^n} \right) \cdot F(x_0) - g(x_0) \right| = N_c^*(g).$$

Hence,

$$N_c^*(g) = \left| \left(\sum_{n=1}^\infty \frac{a_n}{2^n} \right) \cdot F(x_0) - g(x_0) \right| = \left| \sum_{n=1}^\infty \frac{a_n \cdot F(x_0) - g(x_0)}{2^n} \right|$$

$$\leqslant \sum_{n=1}^\infty \frac{|a_n \cdot F(x_0) - g(x_0)|}{2^n} \leqslant \sum_{n=1}^\infty \frac{N_c^*(g)}{2^n} = N_c^*(g).$$

This implies that for all n, $|a_n \cdot F(x_0) - g(x_0)| = N_c^*(g)$ and $a_n \cdot F(x_0) - g(x_0)$ has a constant sign. Hence, $a_n \cdot F(x_0) = a_m \cdot F(x_0)$ for all m, n. Thus, since

$F(x_0) \cdot a$ is a continuous function of a on S and a constant on a dense subset of S, it must be constant on S. Since $0 \in S$, this establishes the result.

Proof of Theorem 3.1.

Sufficiency: Define a function $h(x)$ on $Q_1 \cup Q_2$ as follows:

$$h(x) = \begin{cases} K + \min_{a \in S} a \cdot F(x), & x \in Q_1 \\ -K + \max_{a \in S} a \cdot F(x), & x \in Q_2 . \end{cases}$$

$h(x)$ is well defined; for if $K = N_c^m$, the two definitions coincide on $Q_1 \cap Q_2$.

Since $h(x)$ is a continuous function on a closed subset of X, it is easy to show, using the Tietze Extension Theorem, that it can be extended to a continuous function, $g(x)$, on the whole of X, satisfying

$$-K + \max_{a \in S} a \cdot F(x) \leqslant g(x) \leqslant K + \min_{a \in S} a \cdot F(x).$$

Thus, for $a \in S$, $\| g(x) - a \cdot F(x) \| \leqslant K$.

Since $x_0 \in Q_1 \cup Q_2$, it follows that $| g(x_0) | = K$. Hence, for $a \in S$, $| g(x_0) - a \cdot F(x_0) | = K$. Thus, if $a \in S$, $\| g(x) - a \cdot F(x) \| = K$.

If $d \notin L(S)$, then $d = rb + v$, where $r < 0$, $| b | = 1$, $b \in L(S)^{\perp}$, and $v \in L(S)$. Since $F(Q_1) \cup (-F(Q_2))$ enfolds S, there is a sequence (T_n), $T_n \in F(Q_1) \cup (-F(Q_2))$, such that

$$\frac{T_n^{S,b}}{| T_n^{S,b} |} \to b.$$

But $T_n^{S,b} = T_n^S + T_n^b$. Therefore,

$$\frac{T_n^S + T_n^b}{| T_n^{S,b} |} \to b.$$

This implies that

$$\frac{| T_n^S |}{| T_n^{S,b} |} \to 0,$$

and, hence, that

$$\frac{| T_n^S |}{| T_n^S | + | T_n^b |} \to 0.$$

It follows that

$$\frac{| T_n^S |}{| T_n^b |} \to 0. \tag{3.3}$$

Further,

$$\frac{(T_n{}^S + T_n{}^b) \cdot b}{| T_n^{S,b} |} \to 1. \tag{3.4}$$

Thus, for large n, $T_n{}^b \cdot b = T_n \cdot b > 0$.

There exists an $M > 0$ such that if $a \in S$, $| v - a | < M$.

Thus, by (3.3) and (3.4), there is an m such that

$$\frac{| T_m{}^S |}{| T_m{}^b |} < \frac{-r}{M}$$

and

$$T_m{}^b \cdot b > 0.$$

So, for $a \in S$, $(v - a) \cdot T_m = (v - a) \cdot T_m{}^S \leqslant M \,| T_m{}^S | < -r \,| T_m{}^b | = -rT_m \cdot b$. Therefore, if $a \in S$, $d \cdot T_m = (rb + v) \cdot T_m < a \cdot T_m$.

If $T_m \in F(Q_1)$, there exists an $x_m \in Q_1$ such that $T_m = F(x_m)$. Therefore,

$$d \cdot F(x_m) - g(x_m) < \min_{a \in S} a \cdot F(x_m) - g(x_m) = -K.$$

If $T_m \in -F(Q_2)$, there exists an $x_m \in Q_2$ such that $T_m = -F(x_m)$, and

$$d \cdot F(x_m) - g(x_m) > \max_{a \in S} a \cdot F(x_m) - g(x_m) = K.$$

Thus, $\| d \cdot F(x) - g(x) \| > K$.

Let $d \in L(S)$, $d \notin S$. Then $d = ra^*$, where a^* is a boundary point of S relative to $L(S)$ and $r > 1$.

Since $F(Q_1) \cup (-F(Q_2))$ enfolds S, there exists $b \in L(S)$ and a sequence (T_n), $T_n \in F(Q_1) \cup (-F(Q_2))$, such that

$$b \cdot (a - a^*) \geqslant 0 \qquad \text{for all } a \in S,$$

and

$$\frac{T_n^{S,b}}{| T_n^{S,b} |} \to b.$$

Because 0 is an interior point of S relative to $L(S)$, $b \cdot (0 - a^*) > 0$. Thus, $b \cdot d = b \cdot ra^* < b \cdot a^* = \min_{a \in S} b \cdot a$. Since $\min_{a \in S} b \cdot a$ is a continuous function of b, there exists an n such that

$$\frac{T_n^{S,b}}{| T_n^{S,b} |} \cdot d < \min_{a \in S} \frac{T_n^{S,b}}{| T_n^{S,b} |} \cdot a.$$

Hence, $T_n^{S,b} \cdot d < \min_{a \in S} T_n^{S,b} \cdot a$. But since $b, d \in L(S)$, we have

$$T_n \cdot d < \min_{a \in S} T_n \cdot a.$$

If $T_n \in F(Q_1)$, there is an $x_n \in Q_1$ such that $T_n = F(x_n)$. Hence, $F(x_n) \cdot d - g(x_n) < \min_{a \in S} F(x_n) \cdot a - g(x_n) = -K$.

If $T_n \in -F(Q_2)$, there is an $x_n \in Q_2$ such that $-F(x_n) \cdot d < \min_{a \in S} -F(x_n) \cdot a$, i.e.,

$$d \cdot F(x_n) > -\min_{a \in S} (-F(x_n)) \cdot a = \max_{a \in S} a \cdot F(x_n).$$

Thus,

$$d \cdot F(x_n) - g(x_n) > \max_{a \in S} a \cdot F(x_n) - g(x_n) = K.$$

Hence, $\| d \cdot F(x) - g(x) \| > K$.

Necessity: Because $N_c^*(g) = K$, and because 0 is an interior point of S relative to $L(S)$, we have

$$-K \leqslant -K + \max_{a \in S} a \cdot F(x) \leqslant g(x) \leqslant K + \min_{a \in S} a \cdot F(x) \leqslant K. \quad (3.5)$$

Let

$$Q_1^* = [x \in X: g(x) = K + \min_{a \in S} a \cdot F(x)],$$

$$Q_2^* = [x \in X: g(x) = -K + \max_{a \in S} a \cdot F(x)].$$

We now proceed to define Q_1 and Q_2.

CASE 1: $K > N_c^m$. In this case,

$$K + \min_{a \in S} a \cdot F(x) > -K + \max_{a \in S} a \cdot F(x).$$

Thus, Q_1^* and Q_2^* are disjoint closed sets.

Hence, there are two open sets, P_1 and P_2, such that $Q_1^* \subset P_1$, $Q_2^* \subset P_2$ and $\bar{P}_1 \cap \bar{P}_2 = \varnothing$.

Define $Q_1 = \bar{P}_1$ and $Q_2 = \bar{P}_2$.

CASE 2: $K = N_c^m$. If S consists of the single point 0, then $N_c^m = 0$, $g(x) = 0$, and $Q_1^* = Q_2^* = X$. Define $Q_1 = Q_2 = X$.

If $S \neq 0$, $N_c^m \neq 0$. For otherwise, there would exist a non-zero $a^* \in S$ such that $\max_{a \in S} a \cdot F(x) \equiv a^* \cdot F(x) \equiv 0 \cdot F(x) \equiv \min_{a \in S} a \cdot F(x)$. This contradicts the fact that the components of F are linearly independent.

On $E_F(S) = Q_1^* \cap Q_2^*$, $\frac{1}{2}(\max_{a \in S} a \cdot F(x) - \min_{a \in S} a \cdot F(x)) = N_c^m > 0$. This implies that $F^S(x) \neq 0$ on $E_F(S)$. Hence, $| F^S(x)|$ attains a non-zero minimum on $E_F(S)$. Let ϵ be this minimum.

Now, because X is compact, $|F^S(x)|$ is uniformly continuous on X. Thus, there is an open set $P \supset Q_1^* \cap Q_2^*$ such that $|F^S(x)| > \epsilon/2$ on P.

Q_2^* and $\mathscr{C}P \cap Q_1^*$ are disjoint closed sets. Hence, there is an open set P_1, $(\mathscr{C}P \cap Q_1^*) \subset P_1$, such that $\bar{P}_1 \cap Q_2^* = \varnothing$. Similarly, since $\mathscr{C}P \cap Q_2^*$ and $\bar{P}_1 \cup Q_1^*$ are disjoint closed sets, there is an open set P_2, $(\mathscr{C}P \cap Q_2^*) \subset P_2$, such that $\bar{P}_2 \cap (Q_1^* \cup \bar{P}_1) = \varnothing$.

Define $Q_1 = \bar{P}_1 \cup Q_1^*$ and $Q_2 = \bar{P}_2 \cup Q_2^*$. Then Q_1 and Q_2 are closed and $Q_1 \cap Q_2 = Q_1^* \cap Q_2^* = E_F(S)$. Further, if $F^S(x) = 0$ and $x \in Q_1^*$, then $x \in (\mathscr{C}P \cap Q_1^*) \subset P_1 \subset \bar{P}_1 \subset Q_1$. Thus x is an interior point of Q_1. Similarly, if $x \in Q_2^*$ and $F^S(x) = 0$, x is interior to Q_2.

To summarize: If $K > N_c^m$, then $Q_1 \cap Q_2 = \varnothing$. If $K = N_c^m$, then $Q_1 \cap Q_2 = E_F(S)$. And in any case, if $x \in Q_i^*$ and $F^S(x) = 0$, then x is an interior point of Q_i.

By Lemma (3.2), there exists an x_0 such that $F(x_0) \in L(S)^\perp$. Further, $|g(x_0)| = K$. Thus, $x_0 \in Q_1^* \cup Q_2^* \subset Q_1 \cup Q_2$.

We now need only to show that $F(Q_1) \cup (-F(Q_2))$ enfolds S.

First, let $b \in L(S)^\perp$, $|b| = 1$, and let $\epsilon_n \to 0$, $\epsilon_n > 0$. Since $-\epsilon_n b \notin L(S)$, there exists an $x_n \in X$ such that $|-\epsilon_n b \cdot F(x_n) - g(x_n)| > K$. Thus, for all $a \in S$, either

$$\text{(i)} \quad -\epsilon_n b \cdot F(x_n) - g(x_n) < -K \leqslant a \cdot F(x_n) - g(x_n),$$

or (3.6)

$$\text{(ii)} \quad g(x_n) + (-\epsilon_n)\, b \cdot (-F(x_n)) < -K \leqslant g(x_n) + a \cdot (-F(x_n)).$$

If (i) holds, let $T_n = F(x_n)$. If (ii) holds, let $T_n = -F(x_n)$. So for all $a \in S$, $-\epsilon_n b \cdot T_n < a \cdot T_n$. Hence, $\epsilon_n b \cdot T_n > 0 \cdot T_n = 0$. Thus, $\epsilon_n b \cdot T_n = \epsilon_n |T_n^b|$. Therefore, $\epsilon_n |T_n^b| > -a \cdot T_n = -a \cdot T_n^S$.

Because 0 is an interior point relative to $L(S)$, there is a $\beta > 0$ such that, for all n, $-\beta(T_n^S/|T_n^S|) \in S$, $T_n^S \neq 0$.

Therefore, $\epsilon_n |T_n^b| > \beta |T_n^S|$. (Note that this inequality is also valid if $T_n^S = 0$.) Hence,

$$\frac{|T_n^S|}{|T_n^b|} \to 0$$

and, therefore

$$\frac{|T_n^S|}{|T^{S,b}|} \to 0.$$

Thus,

$$1 = \operatorname{Lim} \frac{|T_n^{S,b}|}{|T_n^{S,b}|} \leqslant \operatorname{Lim} \frac{|T_n^S|}{|T_n^{S,b}|} + \operatorname{Lim} \frac{|T_n^b|}{|T_n^{S,b}|} = \operatorname{Lim} \frac{|T_n^b|}{|T_n^{S,b}|} \leqslant 1.$$

It follows that

$$\operatorname{Lim} \frac{|T_n^{\,b}|}{|T_n^{S,b}|} = 1.$$

As a consequence,

$$\operatorname{Lim} \frac{T_n^{S,b}}{|T_n^{S,b}|} = \operatorname{Lim} \frac{T_n^{S,b}}{|T_n^{\,b}|} = \operatorname{Lim} \frac{T_n^{\,S}}{|T_n^{\,b}|} + \operatorname{Lim} \frac{T_n^{\,b}}{|T_n^{\,b}|} = \operatorname{Lim} \frac{T_n^{\,b}}{|T_n^{\,b}|}$$

and, since $\epsilon_n b \cdot T_n > 0$,

$$\frac{T_n^{\,b}}{|T_n^{\,b}|} = b.$$

Therefore,

$$\frac{T_n^{S,b}}{|T_n^{S,b}|} \to b.$$

It must now be shown that $T_n \in F(Q_1) \cup (-F(Q_2))$.

With no loss, it can be assumed that $x_n \to x^*$. Since $0 \in S$,

$$K \geqslant |\, 0 \cdot F(x^*) - g(x^*)| = \operatorname{Lim} |\, -\epsilon_n b \cdot F(x_n) - g(x_n)| \geqslant K.$$

Thus, $|\, g(x^*)| = K$. Therefore, by (3.5), $x^* \in Q_1{}^* \cup Q_2{}^*$.

If $F^S(x^*) \neq 0$, there is an $a^* \in S$ such that $a^* \cdot F(x^*)$ has the same sign as $-g(x^*)$. But then, $|\, a^* \cdot F(x^*) - g(x^*)| > K$. Hence, $F^S(x^*) = 0$. This implies that x^* is an interior point of $Q_1 \cup Q_2$. Further, if $x^* \in Q_1{}^*$, (3.5) implies that $g(x^*) = K$. Thus, by (3.6), for large n, $T_n = F(x_n)$. So with no loss, it can be assumed that $T_n \in F(Q_1)$. Similarly, if $x^* \in Q_2{}^*$, then for large n, $T_n = -F(x_n)$. So, again, with no loss it can be assumed that $T_n \in -F(Q_2)$.

To complete the proof, we must show that for each boundary point a^* of S relative to $L(S)$, there is a $b \in L(S)$ such that

(1) $b \cdot (a - a^*) \geqslant 0$, for all $a \in S$, and

(2) There is a sequence (T_n), $T_n \in F(Q_1) \cup (-F(Q_2))$, such that

$$\frac{T_n^{S,b}}{|T_n^{S,b}|} = \frac{T_n^{\,S}}{|T_n^{\,S}|} \to b.$$

Let $a_n \to a^*$, $a_n \in L(S)$, $a_n \notin S$. There exists an $x_n \in X$ such that $|\, a_n \cdot F(x_n) - g(x_n)| > K$. For all $a \in S$, either

$$\text{(i)} \quad a_n \cdot F(x_n) - g(x_n) < -K \leqslant a \cdot F(x_n) - g(x_n),$$

or (3.7)

$$\text{(ii)} \quad g(x_n) + a_n \cdot (-F(x_n)) < -K \leqslant g(x_n) + a \cdot (-F(x_n)).$$

If (i) holds, let $T_n{}' = F(x_n)$. If (ii) holds, let $T_n{}' = -F(x_n)$.

So for all $a \in S$, $a_n \cdot T_n' < a \cdot T_n'$. Thus, $a_n \cdot T_n' < 0$. Therefore, $T_n' \neq 0$. Thus, for $a \in S$,

$$(a - a_n) \cdot \frac{T_n'^S}{| T_n'^S |} > 0.$$

With no loss, it can be assumed that $T'^S_n / | T'^S_n |$ converges to $b \in L(S)$, and also that $x_n \to x^*$. Thus, $(a - a^*) \cdot b \geqslant 0$, and, further, $K = | a^* \cdot F(x^*) - g(x^*)|$.

If $a^* \cdot F(x^*) - g(x^*) = -K$, (3.5) implies that $x^* \in Q_1^*$ and (3.7) implies that for large n, $T_n' = F(x_n)$.

If $F^S(x^*) = 0$, x^* is interior to Q_1. We can assume that $T_n' \in F(Q_1)$, for all n. In this case, define $T_n = T_n'$. If $F^S(x^*) \neq 0$, then

$$b = \frac{F^S(x^*)}{| F^S(x^*)|}.$$

For all n, define $T_n = F(x^*) \in F(Q_1)$.

If $a^* \cdot F(x^*) - g(x^*) = K$, then $x^* \in Q_2$ and for large n, $T_n' = -F(x_n)$. If $F^S(x^*) = 0$, we define, as above, $T_n = T_n'$.

If $F^S(x^*) \neq 0$, for all n, let $T_n = -F(x^*) \in -F(Q_2)$.

References

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